Integration Techniques Cheat Sheet

Edexcel FP2

You have seen that integration is useful for finding areas under graphs, as well as volumes of revolutions. Integration can also be used to find lengths and surface areas of lines and 3D solids defined in Cartesian coordinates. We will now explore new techniques for evaluating different families of integrals, whose integrands are parametrised by the index n.

Reduction Formulae

When evaluating integrals such as $\int \sin^2 x\, dx$, you have seen how useful trigonometric identities can be, and when evaluating integrals in the form $\int x^2 e^x\, dx$ how integration by parts can be used. However, as the indices increase, for example to $\int \sin^5 x\, dx$ or $\int x^6 e^x\, dx$, then the calculations can become long and complicated. It can be useful to rewrite these integrals in terms of similar ones of lower powers, forming a relationship called a reduction formula:

- A reduction formula allows a recurrence relationship to be written for an integral $I_n = \int f(x, n)dx$ in terms of related integrals

Reduction formulae are formed by applying the formula for integration by parts: $\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx} dx$

Example 1: Given that $I_n = \int x^n e^x\, dx$, where n is a positive integer, find a reduction formula for I_n. Use the reduction formula obtained to evaluate $\int x^3 e^x\, dx$

Apply the integration by parts formula	$u = x^n, \frac{dv}{dx} = e^x \Rightarrow \frac{du}{dx} = nx^{n-1}, v = e^x$ So, $I_n = x^n e^x - \int e^x n x^{n-1}\, dx$
Take the factor of n out of the integral	$I_n = x^n e^x - n \int e^x x^{n-1}\, dx$
Notice that the integral is the same as the original one but with a power of $n-1$ instead of n. This is denoted I_{n-1}	$I_n = x^n e^x - nI_{n-1}$ Thus we have obtained a reduction formulae for I_n.
To evaluate I_3, substitute 3 into the reduction formula found	$I_3 = x^3 e^x - 3I_2$
Evaluate I_2 by substituting into the formula, and continue until you reach I_0, this can be evaluated using previously seen methods. This is an indefinite integral so remember the $+c$	$I_3 = x^3 e^x - 3(x^2 e^x - 2I_1)$ $I_3 = x^3 e^x - 3x^2 e^x + 6I_1$ $I_3 = x^3 e^x - 3x^2 e^x + 6(xe^x - I_0)$ $I_0 = \int e^x\, dx = e^x + c$ $I_3 = x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + c$

Some questions may require a bit of manipulation in order to be able to apply integration by parts.

Example 2: Find a reduction formula for $I_n = \int_0^{\frac{\pi}{2}} \sec^n x\, dx$

Separate the integrand into $\sec^{n-2} x$ and $\sec^2 x$.	$I_n = \int_0^{\frac{\pi}{2}} \sec^{n-2} x \sec^2 x$ $u = \sec^{n-2}, \frac{dv}{dx} = \sec^2 x \Rightarrow$ $\frac{du}{dx} = (n-2)\sec^{n-2} x, v = \tan x$
Put into the formula for integration by parts	$I_n = \tan x \sec^{n-2} x - \int (n-2)\sec^{n-2} \tan^2 x\, dx$
Use the trig identity $\tan^2 x = \sec^2 x - 1$	$I_n = \tan x \sec^{n-2} x - (n-2)\int \sec^{n-2}(\sec^2 - 1)dx$
Simplify	$I_n = \tan x \sec^{n-2} x - (n-2)\int \sec^n x - \sec^{n-2} x\, dx$ $I_n = \tan x \sec^{n-2} x - (n-2)\int \sec^n x\, dx +$ $(n-2)\int \sec^{n-2}\, dx$ $I_n = \tan x \sec^{n-2} x - (n-2)I_n + (n-2)I_{n-2}$ $I_n + (n-2)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}$ $(n-1)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}$ $I_n = \frac{1}{(n-1)}\tan x \sec^{n-2} x + \frac{(n-2)}{(n-1)}I_{n-2}$

- Although reduction formulae look very complicated, there are only a certain number of types of questions that you can actually be asked, with practise you will recognise the tricks needed, such as using algebraic or trigonometric identities. Often, the question will guide you towards the necessary 'tricks'.
- You may also be asked to compute a reduction formula for a definite integrals, so you must evaluate uv between the given limits, and the answers will be in terms of numerical values and not functions of x.

Arc length

Integration can be used to find the length of a curve between two points on this curve, which is referred to as the arc length. This isn't to be confused with the arc length of a circle- although they refer to the same concepts, in this chapter 'arc length' is used to refer to the length of any continuous part of a curve and is denoted s.

For a curve in Cartesian form, the arc length between points $A(x_A, y_A)$ and $B(x_B, y_B)$ on the graph $y = f(x)$ is given by

$$s = \int_{x_A}^{x_B} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{or} \quad s = \int_{y_A}^{y_B} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

If the equation of the curve is given parametrically, then the arc length between the points $A(f(t_A), g(t_A))$ and $B(f(t_B), g(t_B))$ on the curve with parametric equations $x = f(t), y = g(t)$, then

$$s = \int_{t_A}^{t_B} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

If the curve is given in polar form, the arc length between the half-lines $\theta = \alpha, \theta = \beta$ on the curve with polar equation $r = f(\theta)$, then

$$s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

These formulae are NOT found in the formula booklet- make sure you are familiar with them

Example 3: Find the length of the arc PQ on the curve with equation $y = \frac{1}{4}x^{\frac{3}{2}}$, where the x-coordinates of P and Q are 5 and 15 respectively to 3 d.p.

Choose the appropriate formula. As the curve is given in Cartesian form and in the form $y =$ (so it is easier to find $\frac{dy}{dx}$) we pick the one with $\frac{dy}{dx}$ in.	$s = \int_{x_A}^{x_B} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$
Find $\frac{dy}{dx}$.	$y = \frac{1}{4}x^{\frac{3}{2}}$ $\frac{dy}{dx} = \frac{3}{8}x^{\frac{1}{2}}$
Substitute into the formula	$s = \int_5^{15} \sqrt{1 + (\frac{3}{8}\sqrt{x})^2}\, dx$
Evaluate the integral using substitution	$s = \int_5^{15} \left(1 + \frac{9}{64}x\right)^{\frac{1}{2}} dx$ $s = \left[\frac{(9x+64)^{\frac{3}{2}}}{108}\right]_5^{15}$ $s = 15.456$

Example 4: A particle travels along the curve represented by the equations $x = t^3 - t, y = 2e^{-t^2}, -1.5 \le t \le 1.5$. Assuming the particle travels the length of the curve exactly once, find an integral expression for the length travelled.

Pick the appropriate formula	$s = \int_{t_A}^{t_B} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
Find the derivatives	$x = t^3 - t$ $\frac{dx}{dt} = 3t^2 - 1$ $y = 2e^{-t^2}$ $\frac{dy}{dt} = -4te^{-t^2}$
Substitute into the formula	$s = \int_{-1.5}^{1.5} \sqrt{(3t^2-1)^2 + (-4te^{-t^2})^2}\, dt$
Simplify	$s = \int_{-1.5}^{1.5} \sqrt{9t^4 - 6t^2 + 1 + 16t^2 e^{-2t^2}}\, dt$

Example 5: Find the length of the curve with polar equation $r = e^\theta$ where $0 \le \theta \le \pi$

Pick the appropriate formula	$s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$
Find the derivative	$r = e^\theta$ $\frac{dr}{d\theta} = e^\theta$
Substitute into the formula	$s = \int_0^{\pi} \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta$
	$s = \int_0^{\pi} \sqrt{2}e^\theta\, d\theta$ $s = [\sqrt{2}e^\theta]_0^\pi$ $s = \sqrt{2}(e^\pi - 1)$

Area of a surface of revolution

If a curve C with equation $y = f(x)$ or $x = f(y)$ is rotated 2π radians about the x or y-axis respectively, then it traces out a solid, the area of which is called the surface of revolution, denoted S. (A capital S denotes an area, whereas s denotes an arc length)

For a curve with Cartesian equation $y = f(x)$ between the points (x_A, y_A) and (x_B, y_B) that is rotated 2π radians about the x-axis, the area of the resulting surface of revolution is given by:

$$S = 2\pi \int_{x_A}^{x_B} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

For a curve with Cartesian equation $x = f(y)$ between the points (x_A, y_A) and (x_B, y_B) that is rotated 2π radians about the y-axis, the area of the resulting surface of revolution is given by:

$$S = 2\pi \int_{y_A}^{y_B} x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \quad \text{or} \quad S = 2\pi \int_{x_A}^{x_B} x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If the curve with parametric equations $x = f(t)$ and $y = g(t)$ between the points (x_A, y_A) and (x_B, y_B) is rotated 2π radians about the co-ordinate axes, then the areas of the resulting surfaces of revolution are given by:

Rotation around the x-axis:

$$S = 2\pi \int_{t_A}^{t_B} y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Rotation around the y-axis:

$$S = 2\pi \int_{t_A}^{t_B} x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

If the curve with polar equation $r = f(\theta)$ between the points where $\theta = \alpha$ and $\theta = \beta$ is rotated about the given lines, then the areas of the resulting surfaces of revolution are given by:

Rotation about the initial line, $\theta = 0$:

$$S = 2\pi \int_{\alpha}^{\beta} r\sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Rotation about the line $\theta = \pm\frac{\pi}{2}$:

$$S = 2\pi \int_{\alpha}^{\beta} r\cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example 6: A parabola has the equation $y^2 = 12x$. The arc between the values of $x = 0$ and $x = 3$ is rotated 2π radians about the x-axis. Find the area of the curved surface of the solid produced to 2 d.p.

Select the appropriate formula.	$S = 2\pi \int_{x_A}^{x_B} y \sqrt{1 + (\frac{dy}{dx})^2}\, dx$
Find the derivative.	$y^2 = 12x$ $2y\frac{dy}{dx} = 12$ $\frac{dy}{dx} = \frac{6}{y}$
Substitute into the formula and evaluate the integral.	$S = 2\pi \int_0^3 y \sqrt{1 + \frac{36}{y^2}}\, dx$ $S = 2\pi \int_0^3 y \sqrt{\frac{y^2 + 36}{y^2}}\, dx$ $S = 2\pi \int_0^3 y \sqrt{\frac{12x + 36}{y^2}}\, dx$ $S = 2\pi \int_0^3 (12x + 36)^{\frac{1}{2}}\, dx$ $S = 2\pi \left[\frac{(12x+36)^{\frac{3}{2}}}{18}\right]_0^3$ $S = 2\pi(21.941)$ $S = 137.86$